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THE PROBLEM OF THE DIFFRACTION OF INTERNAL WAVES AT THE EDGE OF A SEMI-INFINITE FILM*

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In a continuation of the research described in /1-4/ on the diffraction of waves, described by the Klein-Gordon equation, the diffraction of external waves at the boundary of a semi-infinity film situated on the surface of a stratified liquid is considered. Among the many papers devoted to the scattering of acoustic waves by rectilinear objects we mention /5-7/. The need to take into account the properties of the surface covering the liquid led to a study of the boundary value problem for the Helmholtz equation with boundary conditions containing higher-order derivatives than the equation itself. Consideration of the surface tension of a semi-infinite film leads to a similar situation. However, in this case the propagation of the waves is described by an equation of the hyperbolic and not the elliptic type.

1. To study two-dimensional motions of an incompatible ideal liquid we will introduce a Cartesian system of coordinates $\{x, O, z\}$. Consider an infinite plane layer $Q = \{(x, z): -\infty < x < \infty, -h < z < 0\}$ of a stratified liquid, bounded from below (for $z = -h$) by a solid bottom. Above (where $z = 0$) the boundary of the liquid consists of two parts; for $x < 0$ the surface of the liquid is free, and for $x \geqslant 0$ the liquid is covered by a thin film having a surface tension σ. The density of the liquid in the unperturbed state has the distribution $\rho_0(z) = \rho_0 e^{-2\beta z}$, $\beta > 0$.

The small oscillations of the liquid are described by the following system of equations /8/:

$$\begin{gathered}\rho_0(z)\partial \mathbf{V}/\partial t + \nabla p + \mathbf{e}_z\rho_1 g = 0\\ \partial/\partial t\rho_1 + (\mathbf{e}_z, \mathbf{V})\rho_0'(z) = 0, \ \operatorname{div}\mathbf{V} = 0\end{gathered} \tag{1.1}$$

where $\mathbf{V} = \{v_1, v_2\}$ is the vector of the velocity of the liquid particles, ρ_1 is the change in the density due to motions of the liquid, p is the dynamic pressure, $\mathbf{e}_z$ is the unit vector of the Oz axis, and g is the acceleration due to gravity.

If we introduce the stream function Ψ using the formulas $v_1 = \Psi_z$, $v_2 = -\Psi_x$ and then the function $u = \Psi e^{-\beta z}$, the integration of system (1.1) can be reduced to solving the equation

$$\partial^2/\partial t^2 [\Delta_2 u - \beta^2 u] + \omega_0^2 u_{xx} = 0 \tag{1.2}$$

where Δ_2 is the Laplace operator with respect to x and z and $\omega_0^2 = 2\beta g$ is the square of the Brent-Viaisial frequency.

For steady-state wave motion, which depends on time as $e^{-i\omega t}$, and $\omega < \omega_0$, Eq.(1.2) can be written as the Klein-Gordon equation

$$u_{zz} - \beta^2 u = \frac{1}{a^2} u_{xx}, \quad \frac{1}{a^2} = \frac{\omega_0^2}{\omega^2} - 1 > 0 \tag{1.3}$$

The condition for the solid bottom to be impenetrable and the boundary condition on the free surface /2/ have the form

$$u = 0, \ z = -h, \ x \in R^1 \tag{1.4}$$

$$u_z + \beta u + (g/\omega^2)u_{xx} = 0, \ z = 0, \ x < 0 \tag{1.5}$$

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The condition when $z=0,\ x>0$ can be derived from the relations

$$\frac{\partial\zeta}{\partial t}=v_z,\quad p=\rho_0 g\zeta-\sigma\frac{\partial^2\zeta}{\partial x^2},\quad z=0,\quad x>0 \tag{1.6}$$

and Eqs.(1.1) ($\zeta(x,t)$ is the vertical displacement of the film). Eliminating the function ζ from (1.6) we can write the following boundary condition for the stream function Ψ:

$$\rho_0\frac{\partial^3\Psi}{\partial z\,\partial t^2}=\rho_0 g\quad\frac{\partial^2\Psi}{\partial x^2}-\sigma\frac{\partial^4\Psi}{\partial x^4},\quad z=0,\quad x>0$$

Bearing the chosen time dependence in mind, we can rewrite the latter relation in terms of the function u:

$$\frac{\partial u}{\partial z}+\beta u+\frac{g}{\omega^2}\frac{\partial^2 u}{\partial x^2}-\frac{\sigma}{\rho_0\omega^2}\frac{\partial^4 u}{\partial x^4}=0,\quad z=0,\quad x>0 \tag{1.7}$$

At the edge of the semi-infinite film the boundary-contact condition /5/ must be satisfied, corresponding to the fact that there is no concentrated force on it: $\partial\zeta(+0)/\partial x$. Since it follows from the first relation of (1.6) that $\zeta(x)=(i\omega)^{-1}\times u_x(x,0)$, this condition can be rewritten in the form

$$\lim_{x\to+0}\frac{\partial^2 u}{\partial x^2}(x,0)=0 \tag{1.8}$$

Suppose that in the region $x<0,\ -h<z<0$ a wave $u_N=\sin b_N(z+h)e^{ik_N x}$ propagates from infinity to the side of the semi-infinite film in a horizontal direction (one of the normal waves of the layer of liquid bounded above by the free surface and below by the solid bottom). Here $k_N,\ b_N>0$ are the components of the wave vector, where b_N is one of the roots of the dispersion equation /3/ $b\operatorname{ctg} bh=-\beta+ga^2(b^2+\beta^2)/\omega^2$ and $k_N{}^2=a^2(b_N{}^2+\beta^2)$ according to (1.3). It is required to investigate the wave motions excited by this wave in the liquid.

If we represent the total wave field u_Σ in the form $u_\Sigma=u+u_N$, the problem of determining the function u takes the following form: it is required to find everywhere the bounded function u which satisfies Eq.(1.3) in the region Q in the generalized sense, as well as the boundary conditions (1.4) and (1.5) and the conditions, which follow from (1.7) and (1.8),

$$\frac{\partial u}{\partial z}+\beta u+\frac{g}{\omega^2}\frac{\partial^2 u}{\partial x^2}-\frac{\sigma}{\rho_0\omega^2}\frac{\partial^4 u}{\partial x^4}=A_N e^{ik_N x},\quad z=0,\quad x>0$$
$$\lim_{x\to+0}\frac{\partial^2}{\partial x^2}(u+u_N)|_{z=0}=0,\quad A_N=\frac{\sigma k_N{}^4\sin b_N h}{\rho_0\omega^2}$$

We will require that the function u and its first derivatives should be bounded at the origin of coordinates, while the higher-order derivatives at this point will have a singularity no higher than that indicated by the following:

$$\left|\frac{\partial^2 u}{\partial x^2}\right|\leqslant Cx^{-1/2},\quad \left|\frac{\partial^3 u}{\partial x^3}\right|\leqslant Cx^{-3/2},\quad \left|\frac{\partial^4 u}{\partial x^4}\right|\leqslant Cx^{-5/2},\quad x\to 0,\quad z=0$$

These conditions follow from the fact that the energy flow through an aribtrary closed surface surrounding the edge of the obstacle, is zero.

We will formulate the radiation conditions in the form of a requirement such that the wave which arise as a result of diffraction carry away energy to infinity.

2. The solution of the problem which satisfies all the requirements formulated above can be constructed by the Wiener-Hopf method /9/, and has the form

$$u=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{iA_N\Omega_-(-k_N)}{\alpha+k_N}+C_0\right]\frac{\sin\gamma(z+h)/a}{\gamma h/a}\frac{\Omega_+(\alpha)}{\Omega_1(\alpha)}e^{-i\alpha x}\,d\alpha \tag{2.1}$$

The integration in (2.1) is carried out along the real axis of the plane α, circumventing the singular points of the integrand, where the negative singular points are circumvented from above and the positive ones from below. The branch of the function $\gamma(\alpha)=(\alpha^2-a^2\beta^2)^{1/2}$ is chosen as follows. We make a cut in the plane α connecting the points $-a\beta$ and $a\beta$ through an infinitely removed point, and which goes vertically upwards in the upper half-plane and vertically downwards in the lower half-plane. We distinguish the branch of the function $\gamma(\alpha)$ for which $\gamma(0)=-ia\beta$.

When using the Wiener-Hopf method, the need arises to investigate the functions

$$\Omega_1(\alpha)=\cos H-(BH^2+C)H^{-1}\sin H \tag{2.2}$$
$$\Omega_2(\alpha)=\cos H-\{A\,[H^2+(\beta h)^2]+BH^2+C\}H^{-1}\sin H \tag{2.3}$$
$$\left(A=\frac{\sigma(1+a^2)\,a^2}{h\,(\rho_0 gh)\,2\beta h},\ B=\frac{1+a^2}{2\beta h},\ C=\frac{\beta h}{2}(a^2-1),\ H=\frac{\gamma h}{a}\right)$$

The function $\Omega_+(\alpha)$ appears on factorizing the function $\Omega(\alpha) = \Omega_1(\alpha)/\Omega_2(\alpha)$, and we have for it the estimate $\Omega_+(\alpha) = O(|\alpha|^{-1})$ when $|\alpha| \to \infty$, $\mathrm{Im}\,\alpha \geqslant 0$.

It can be shown that the function (2.3) has a denumerable set of zeros $\pm\alpha_n$ $(n = 1, 2, \ldots)$, for which when $n \gg 1$, the following asymptotic formulas hold:

$$\alpha_n = \pi a(n-1)/h + O(1/n), \quad 0 < a < a_p \tag{2.4}$$

$$\alpha_n = \pi an/h + O(1/n), \quad a > a_p \tag{2.5}$$

$$a_p^2 = \{[(k+\beta h/2)^2 + 4k(\beta h/2 + 1)]^{1/2} - (k + \beta h/2)\}/2k$$

$$k = \sigma\beta(\beta h)^2/(2\rho_0 gh)$$

Note that when $a > a_p$, real roots $\pm\alpha_p$ appear in the function (2.3) corresponding to a purely imaginary value of γ_p. In fact, putting $\gamma h/a = -iy$ and rewriting the equation $\Omega_2(\alpha) = 0$ in the form

$$y\,\mathrm{cth}\,y = A[y^2 - (\beta h)^2]^2 - By^2 + C \tag{2.6}$$

it can be shown that when $a > a_p$ it has a positive root $y_p < \beta h$, to which there correspond the roots $\pm\alpha_p$, where $\alpha_p = (a/h)[(\beta h)^2 - y_p^2]^{1/2}$. In addition, for any $a > 0$, the function (2.3) has purely imaginary roots $\pm\bar{\alpha}$, $\bar{\alpha} = -i(a/h)[\bar{y}^2 - (\beta h)^2]^{1/2}$ ($\bar{y} > \beta h$ is the root of Eq.(2.6)), corresponding to the purely imaginary value $\bar{\gamma}$.

The function (2.2), as shown in /3/, has a denumerable set of zeros $\pm\alpha_n^1$ $(n = 1, 2, \ldots)$, defined by an asymptotic formula of the form (2.4), when $0 < a < a_s$, and the form (2.5) when $a > a_s$, where $a_s^2 = 1 + 2/(\beta h)$. It can be shown that $a_p > a_s$ for any $\beta, h, k > 0$. In addition, when $a > a_s$, real roots $\pm\alpha_0$ appear in the function (2.2), corresponding to a purely imaginary value of γ_0.

It follows from the above and Hadamard's theorem on the expansion of an integral function in factors, that the factorization of the function $\Omega(\alpha)$ has the form

$$\Omega(\alpha) = \Omega_+(\alpha)\Omega_-(\alpha), \quad \Omega_-(\alpha) = \Omega_+(-\alpha)$$

$$\Omega_+(\alpha) = \begin{cases} \Pi(\alpha)/(1+\alpha/\bar{\alpha}), & 0 < a < a_p \\ \Pi(\alpha)/[(1+\alpha/\alpha_p)(1+\alpha/\bar{\alpha})], & a_p < a < a_s \\ \Pi(\alpha)(1+\alpha/\alpha_0)/[(1+\alpha/\alpha_p)(1+\alpha/\bar{\alpha})], & a > a_s \end{cases}$$

$$\Pi(\alpha) = \prod_{n=1}^{\infty} \frac{1+\alpha/\alpha_n^1}{1+\alpha/\alpha_n}$$

where $\pm\alpha_0$, $\pm\alpha_n^1$ are the roots of the function (2.2).

We will represent the solution obtained in another form. We will split the region Q into two regions: $Q_1 = \{(x,z): x < 0, -h < z < 0\}$ and $Q_2 = \{(x,z): x > 0, -h < z < 0\}$. Applying Cauchy's theorem to integral (2.1), we obtain the following representation for the total wave field in the region Q_2:

$$\begin{aligned} u_\Sigma(x,z) = {} & D(-\alpha_p)\frac{\mathrm{sh}\,|\gamma_p|(z+h)/a}{|\gamma_p|h/a}\exp(i\alpha_p x) + \\ & D(-\bar{\alpha})\frac{\mathrm{sh}\,|\bar{\gamma}|(z+h)/a}{|\bar{\gamma}|h/a}\exp(-|\bar{\alpha}|x) + \\ & \sum_{n=1}^{\infty} D(-\alpha_n)\frac{\sin\gamma_n(z+h)/a}{\gamma_n h/a}\exp(i\alpha_n x), \quad \gamma_n = \gamma(\alpha_n) \end{aligned} \tag{2.7}$$

$$E(\alpha) = \frac{\sigma k_N^4 \sin b_N h\,\Omega_-(-k_N)}{\rho_0\omega^2(k_N+\alpha)} - iC_0, \quad D(\alpha) = \frac{E(\alpha)}{\Omega_-(\alpha)\,\Omega_2'(\alpha)}$$

The constant C_0 is found from the boundary-contact condition (1.8). In view of the complexity of the corresponding expression it will not be given here.

We similarly have in the region Q_1

$$\begin{aligned} u_\Sigma(x,z) = {} & \sin b_N(z+h)\exp(ik_N x) + \\ & F(\alpha_0)\frac{\mathrm{sh}\,|\bar{\gamma}_0|(z+h)/a}{|\gamma_0|h/a}\exp(-i\alpha_0 x) + \\ & \sum_{n=1}^{\infty} F(\alpha_n^1)\frac{\sin\gamma_n^1(z+h)/a}{\gamma_n^1 h/a}\exp(-i\alpha_n^1 x) \end{aligned} \tag{2.8}$$

$$\gamma_n^1 = \gamma(\alpha_n^1), \quad F(\alpha) = -E(\alpha)\frac{\Omega_+(\alpha)}{\Omega_1'(\alpha)}$$

Expressions (2.7) and (2.8) can be written for the case $a > a_s$. When $0 < a < a_s$ we must omit the second term in Eq.(2.8), and when $0 < a < a_p$, in addition, we must omit the first term in Eq.(2.7).

As an analysis of Eqs.(2.7) and (2.8) shows, the solution constructed is continuously differentiable in the regions Q_2 and Q_1. The second derivatives of the function u_Σ have a logarithmic singularity in the region of the edge of the semi-infinite film. This singularity "propagates" along the characteristics of the equation considered, and is "reflected" from the boundaries of the liquid in accordance with the laws of geometrical optics. In fact, as was done in /3/, it can be shown that the second derivatives of u_Σ have a logarithmic singularity along the sections of the characteristics of Eq.(1.3), defined by the expressions $z - ax = 2mh,\ z + ax = -2(m+1)h,\ m = 0, 1, \ldots$. The third derivatives of the solution have a singularity of the order of the inverse distance, while the fourth derivatives have a singularity of the order of the square of the inverse distance to the corresponding section of the characteristic.

Note that when $z = 0$, there is an improvement in the convergence of the series (2.7) and (2.8). As a result the function $\partial^2 u_\Sigma/\partial x^2$ becomes bounded when $z = 0$. In order to show this we will differentiate series (2.7) and (2.8) twice with respect to x and put $z = 0$ in the expressions obtained. Bearing in mind that $\sin \gamma_n h/a = (-1)^n O(1/n)$ when $n \gg 1$, according to (2.4) and (2.5) (a similar relation holds for $\sin \gamma_n{}^1 h/a$) we conclude that the series considered converge absolutely and uniformly with respect to x. It can similarly be shown that the function $\partial^3 u_\Sigma(x, 0)/\partial x^3$ at the points $x_m = 2mh/a,\ m = 0, \pm 1, \ldots$ has a logarithmic singularity, while the function $\partial^4 u_\Sigma(x, 0)/\partial x^4$ has a singularity of the order of the inverse distance.

The solution constructed satisfies Eq.(1.3) in the generalized sense. The boundary condition (1.5) and the boundary-contact condition (1.8) are satisfied in the classical sense, while boundary conditions (1.7) is satisfied in the generalized sense.

3. We will discuss the results obtained. As can be seen from (2.7) the incident wave, having been scattered at the edge of the semi-infinite film, excites an infinite number of normal internal waves in the waveguide formed by the solid bottom and the semi-infinite film. These waves transfer energy in the positive direction of the Ox axis without attenuation. Their amplitudes decrease as the wave number n increases as C_1/n^3 for $n \gg 1$, while the wavelengths are $\lambda_n = 2\pi/\alpha_n$, where α_n are determined by the asymptotic form of (2.4) and (2.5). When $a > a_p$, in Eq.(2.7) we will have the first term, which describes a flexure wave of the film and a surface wave in the liquid under the film. The amplitude of this wave is a maximum on the surface and decreases exponentially in the direction of the bottom. The second term in (2.7) has a non-wave form, which decreases exponentially with respect to x and z with distance from the edge.

In the region Q_1 when $0 < a < a_s$ there is an infinite number of reflected normal internal waves, whose energy propagates in the negative direction of the Ox axis without attenuation (see (2.8)). The amplitudes of these waves decrease as the wave number increases as C_2/n^3 for $n \gg 1$, while the wavelengths are described in /3/.

When $a > a_s$ the surface wave, represented by the second term in Eq.(2.8), also participates in the formation of the wave pattern. This wave appears for higher excitation frequencies than the flexure wave of the film.

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